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A note on the stability of single frequency laser diode models

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Abstract
A formal proof of the stability of a widely applicable form of the laser diode rate equations is presented. The conditions under which this proof holds provide guidance to designers and modellers of laser diodes in the identification of ranges of parameter values that preserve stability. The proof requires that there is some non-zero threshold for gain in order to guarantee stability. Laser models incorporating Purcell enhancement factors are found to be stable for all physically allowed values of the factor.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

An extremely desirable feature of laser diodes for applications in high bit rate optical communications is the ability to modulate the light output at high frequency and with high linearity by the addition of a modulated current to the bias current. There is a continuing research and development effort to produce single mode, low chirp, linear response laser diodes with high quantum efficiency and low threshold [1, 2]. Some of that research effort has been based on a consideration of the physical and mathematical models used to simulate and understand the behaviour of these devices [3–5]. Identification of those parameters in the model that limit the modulation speed or that introduce chirp or nonlinearities combined with an understanding of the physics and structure of the devices which are represented by those parameters guides the design and manufacture of improved devices. An important question [6] that arises as a result is that of stability: over what range of parameter values is the model which simulates a laser stable? If the model accurately portrays the behaviour of the laser diode this range will also predict the range over which the physical laser diode is stable.

Research into the possibility of engineering the dynamics of laser diodes by the use of the Purcell effect has gained momentum due to recent experimental progress [2, 7]. A simple explanation of how the Purcell effect may be used to modify laser dynamics is that since the rate at which the carrier density within the cavity is changed is controlled by the carrier spontaneous emission rate, a laser diode with faster response time can be engineered by increasing the spontaneous emission rate through quantum electrodynamic effects via design of the cavity. This enhancement gives rise to questions of stability.

Unstable operation is desirable for some applications, particularly those involved in the study or application of chaotic dynamics [8–11]. In this case the ability to identify regions where unstable behaviour is likely is also valuable. Such dynamics can be achieved by reflecting some of the output light back into the cavity, thereby modifying the system dynamics via time-delayed feedback [12, 13]. Such time-delayed systems are not considered here.

Another application of modulated laser diodes for which it is desirable to understand stability is their use as pump sources for Raman amplification [14, 15]. Stable operation at relatively high speed modulation for very high power diodes is required for transient suppression in Raman amplifiers, particularly for the case of co-propagating signal and pump operation. The design of high quality control systems relies on an understanding of these stability properties.

The stability of laser diode models is thus of particular interest in three main operational contexts, which may be summarized as follows:

(a) static operation, due to a static drive current [16, 17];
(b) quasi-equilibrium modulation, in which the laser is modulated at a rate well below its dynamic response time [18];
(c) dynamic high speed modulation [17].

This paper focuses on stability issues associated with (a) and (b), in which the laser diode is modelled via a reasonably general mathematical form for the coupled nonlinear rate equations, and restricted via specific conditions imposed upon the attendant functions and parameters appearing in those equations. The main result of the paper allows for the conclusion of two specific types of stable behaviour, both of which are described below:

- *'Input-to-state stable' (ISS).* A laser diode or model is ISS if, given any fixed input current limit, all current input waveforms inviolate of that limit give rise to photon and carrier density time functions that are also ultimately limited, with the limits obtained being independent of the initial photon and carrier densities, but scaling with the aforementioned input current limit. (ISS [19] generalizes global asymptotic stability (GAS) [20, 21] to non-autonomous dynamical systems with $\mathcal{L}_\infty$ bounded inputs.)
- *'Stationary-point stable' (SPS).* A laser diode or model is SPS if, given a fixed and constant input current limit and any initial photon and carrier density, the time evolution of the photon and carrier densities for any constant input current (satisfying the aforementioned current limit) converge to a unique stationary point in phase space. (SPS generalizes GAS to capture the stability of any stationary point arising from the application of a constant input. SPS is a special case of the Cauchy gain property [22].)

Whilst it is widely recognized that laser diodes are in practice both ISS and SPS over a wide range of operating conditions, a formal proof of this for a sufficiently general model has not been published [23, 24]. (We note that in previous work by Mena *et al*, where a number of specific models are examined, the possible existence of limit cycles in the dynamics is not excluded [23].) The generality of the main result means that it provides a simple test for stability in the sense of ISS and SPS for existing and new candidate laser diode models.

Section 2 introduces various models of laser diodes. A generic model which incorporates the most important features of the typical model is presented in section 3. In section 4, the main result of the paper is stated along with the conditions under which the result is valid. The proof of the stability of this generic model is given in section 5. A summary of the conclusions is the subject of section 6.

2. Specific laser diode models

2.1. Model 1

Laser diodes are typically modelled by a set of coupled nonlinear rate equations for the carrier density $N(t)$ and the photon density $S(t)$ within the laser cavity, a particular example of which is

$$\dot{S}(t) = \frac{g_0\Gamma(N(t)-N_t)}{1+\epsilon S(t)}S(t) - \frac{S(t)}{\tau_{\text{ph}}} + \beta\frac{\Gamma N(t)}{\tau_{\text{sp}}}, \tag{1}$$

$$\dot{N}(t) = \frac{I(t)}{eV} - \frac{g_0(N(t)-N_t)}{1+\epsilon S(t)}S(t) - \frac{N(t)}{\tau_{\text{sp}}}, \tag{2}$$

where g_0 is the gain coefficient, Γ is the modal confinement factor, N_t is the carrier density required for transparency, ϵ is a saturation parameter, τ_{ph} is the average time spent by a photon in the cavity, β is the fraction of spontaneous emission into the lasing mode, τ_{sp} is the spontaneous emission lifetime, V is the volume of the active region, e is the electronic charge and $I(t)$ is the modulated current. Further details can be found in [25, 26] (for example).

The phase of the output light field can be determined from the spectator equation

$$\dot{\varphi}(t) = \frac{\alpha}{2}g_0\Gamma(N(t)-N_t)$$

which models the changing phase of the light field due to the changing refractive index of the cavity with carrier density. The amplitude of the light field can be determined by taking into account the rate at which photons escape the cavity. Here α is the linewidth enhancement factor.

Model 1 is defined by (1) and (2).

2.2. Model 2

Variations on these equations exist which introduce additional physics or which treat the physics in alternative ways. For example, Lau *et al* investigated the behaviour of a set of differential equations to test the virtue of using the Purcell effect to enhance the dynamics of a quantum well laser diode [2]. The differential equations used in that work are reproduced here as

$$\dot{S}(t) = \frac{\Gamma g_0 \ln\left(\frac{N(t)}{N_t}\right)S(t)}{1+\epsilon S(t)} - \frac{S(t)}{\tau_{\text{ph}}} + \beta F\Gamma B N(t)^2, \tag{3}$$

$$\dot{N}(t) = J(t) - \frac{g_0 \ln\left(\frac{N(t)}{N_t}\right)S(t)}{1+\epsilon S(t)} - (\beta F + 1 - \beta)BN(t)^2 - CN(t)^3. \tag{4}$$

Here, $J(t)$ is the injection current density, the gain involves a logarithm of the carrier density and there are more complicated relaxation terms characterized by B and C. The solutions to these equations are integrated over the active region. In this example the gain term is proportional to $\ln(N(t)/N_t)$ which may be expanded to lowest order as $N(t) - N_t$ to give gain terms in the same form as model 1. Bimolecular $(N(t)^2)$ and Auger $(N(t)^3)$ relaxation terms are used in place of the simple relaxation term above. The factor of greatest interest to Lau *et al* was the Purcell factor F which is used to characterize the change of the spontaneous emission rate due to the details of the cavity.

Model 2 is defined by (3) and (4).

2.3. Model 3

Another variant was used by Ngai and Liu [6] to demonstrate the onset of chaotic behaviour in a modulated laser diode. The rate equations used in that work may be written as

$$\dot{S}(t) = \frac{\Gamma g_0 (N(t) - N_t) S(t)}{1 + \epsilon S(t)} - \frac{S(t)}{\tau_{\text{ph}}} + \beta\Gamma C_1 N(t)^2, \tag{5}$$

$$\dot{N}(t) = \frac{I(t)}{eV} - \frac{g_0 (N(t) - N_t) S(t)}{1 + \epsilon S(t)} - C_0 N(t) - C_1 N(t)^2 - C_2 N(t)^3 - D_r N(t)^{\frac{11}{2}}. \tag{6}$$

Here the parameters C_0, C_1, C_2 and D_r are the carrier recombination coefficient, the carrier bimolecular recombination coefficient, the Auger recombination coefficient and the carrier leakage coefficient, respectively. Details of typical values for these parameters can be found in [6] and references therein.

Model 3 is defined by (5) and (6).

2.4. Model 4

A common alternative to model 1 follows by the substitution of an alternative saturation term. In particular, the term $(1 + \epsilon S)^{-1}$ is replaced with $1 - \epsilon S$. It is evident that both of these forms are low-order approximations to some unknown function of S and possibly N that is the *true* saturation factor. A set of equations identical to those in (1) and (2) with the saturation factor $1 - \epsilon S$ has been published in [3, 5, 27]. The physical effects of these saturation models have been investigated in [28]. It may be noted that the aforementioned forms are lower order approximations to the form $(1 + \epsilon S)^{-\frac{1}{2}}$ derived by Agrawal [29].

Model 4 is defined by (1) and (2) with the saturation factor $(1+\epsilon S)^{-1}$ replaced by $(1-\epsilon S)$.

2.5. Model 5

More recently Ahmed and El-Lafi included a saturation factor similar to $1 - \epsilon S$ but which included a dependence on the carrier density [24]. The equations used by them are written here as

$$\dot{S}(t) = (G(t) - G_{th})S(t) + \frac{C}{\tau_{\text{sp}}} N(t), \tag{7}$$

$$\dot{N}(t) = \frac{I(t)}{e} - A(t)S(t) - \frac{N(t)}{\tau_{\text{sp}}}, \tag{8}$$

where

$$G(t) := A(t) - B(t)S(t),$$
$$A(t) := \frac{g_0}{V}(N(t) - N_t),$$
$$B(t) := B_0(N(t) - N_t).$$

Here B plays a similar role to that played by ϵ in model 4 with the additional complication of dependence on the carrier density. Further details and typical values for the parameters in (7) and (8) may be found in [24]. While (7) and (8) have many similarities in appearance to the other laser diode rate equations, they are in fact quite different, that difference to be elaborated on below.

3. A generic laser diode model

3.1. Generic model

We can generalize models 1–3 via the following pair of differential equations:

$$\dot{x}_1 = \theta(x_1)\phi(x_2) - \Upsilon_1(x_1) + \zeta\psi(x_2), \tag{9}$$

$$\dot{x}_2 = -\theta(x_1)\phi(x_2) - \Upsilon_2(x_2) - \psi(x_2) + u. \tag{10}$$

Here, x_1 and x_2 are scaled variables representing the photon density and carrier density respectively, $\theta, \phi, \Upsilon_1, \Upsilon_2$ and ψ are functions of x_1 or x_2 which represent respectively generalized notions of saturation, gain, photon relaxation, non-radiative carrier relaxation and radiative carrier relaxation as appropriate, and u is a scaled current density. ζ represents the effective spontaneous emission factor which captures the difference between the rate of carrier relaxation and the rate of spontaneous emission into the lasing mode. The major limitation of this form is the requirement that the gain terms factor into a function which depends upon the carrier density only, and a function which depends on the photon density only.

The functions and parameters appearing in the differential equations (9) and (10) are restricted to satisfy a number of technical assumptions. These assumptions represent sufficient conditions for stability of the generic model, in the sense of the main result presented in the following section. In order to state these assumptions, a number of function classes are utilized, in particular, $\mathcal{K}_0$, $\mathcal{K}$ and $\mathcal{K}_\infty$. These are defined as follows: a scalar-valued function defined on $[0, \infty)$ is of class $\mathcal{K}_0$ if it is continuous, zero at zero, and non-decreasing, of class $\mathcal{K}$ if it is class $\mathcal{K}_0$ and strictly increasing, and of class $\mathcal{K}_\infty$ if it is class $\mathcal{K}$ and radially unbounded. (Hence, $\mathcal{K}_0 \supset \mathcal{K} \supset \mathcal{K}_\infty$.) In the statement of the main result, the class $\mathcal{KL}$ of scalar-valued functions on $[0, \infty)^2$ is of interest. There,

$$\mathcal{KL} := \left\{ \beta : \mathbf{R}_{\geqslant 0} \times \mathbf{R}_{\geqslant 0} \mapsto \mathbf{R}_{\geqslant 0} \;\middle|\; \begin{array}{c} \beta(\cdot, t) \in \mathcal{K} \text{ for each fixed } t \geqslant 0 \\ \beta(s, \cdot) \text{ is decreasing for each fixed } s \geqslant 0 \\ \text{and } \lim_{t\to\infty} \beta(s, t) = 0 \end{array} \right\}.$$

Elsewhere, $\mathbf{R}^n_{\geqslant 0}$ denotes the closed positive orthant of $\mathbf{R}^n$. $\mathbf{R}^n_{>0}$ denotes the corresponding open set. With this notation in place, the technical assumptions restricting the functions and parameters of (9) and (10) are stated in table 1.

The *generic model* is thus defined by the differential equations (9) and (10) as restricted by assumptions (11)–(19) of table 1. From the point of view of notation, it is convenient to

Table 1. Assumptions restricting rate equations (9) and (10).

Function or parameter	Assumption	
θ, Υ_1 (saturation, photon relaxation)	$\theta \in \mathcal{K} \cap C^1(\mathbf{R}_{\geqslant 0})$	(11)
	$\Upsilon_1 \in \mathcal{K}_\infty \cap C^1(\mathbf{R}_{\geqslant 0})$	(12)
	$(\Upsilon_1/\theta)'(s) > 0 \; \forall s \in \mathbf{R}_{>0}$	(13)
	$L := \lim_{s \to 0^+} (\Upsilon_1(s)/\theta(s)) > 0$ exists	(14)
ϕ (gain)	$\phi(\cdot + s) - \phi(s) \in \mathcal{K}_\infty \cap C^1(\mathbf{R}_{\geqslant 0}) \; \forall s \in \mathbf{R}_{>0}$	(15)
	$c := \phi^{-1}(0) > 0$ exists	(16)
Υ_2 (non-radiative carrier relaxation)	$\Upsilon_2 \in \mathcal{K}_0 \cap C^1(\mathbf{R}_{\geqslant 0})$	(17)
ψ (radiative carrier relaxation)	$\psi \in \mathcal{K}_\infty \cap C^1(\mathbf{R}_{\geqslant 0})$	(18)
ζ (effective spontaneous emission factor)	$\zeta \in (0, 1)$	(19)

denote this generic model by the symbol $\mathcal{G}$, and to write an integrated phase-space trajectory explicitly as the function $x : \mathbf{R}_{\geqslant 0} \mapsto \mathbf{R}^2$, where

$$x = \mathcal{G}_{x_\circ}[u] := \begin{bmatrix} x_1(\cdot) \\ x_2(\cdot) \end{bmatrix} \left| \begin{array}{c} \text{equations (9) and (10) hold, subject} \\ \text{to assumptions (11)–(19), with} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = x_\circ \in \mathbf{R}^2_{>0} \\ \text{with } u(t) \text{ given, for all } t \geqslant 0 \end{array} \right. \tag{20}$$

where $x_\circ \in \mathbf{R}^2_{>0}$ and $u : \mathbf{R}_{\geqslant 0} \mapsto \mathbf{R}_{>0}$ denote respectively an initial density vector and a forcing input function.

It is important to emphasize that the functions and parameters $\theta, \phi, \Upsilon_1, \Upsilon_2, \psi$ and ζ that appear in the generic model (20) represent generalized notions of saturation, gain, photon relaxation, non-radiative carrier relaxation, effective spontaneous emission factor and radiative carrier relaxation. (Here, by 'radiative' we mean radiating into the lasing mode.) Other models may be of the form of (20), but may include terms that do not carry these specific physical meanings. However, this does not affect the applicability of the main result that is presented.

3.2. Qualitative discussion of assumptions (11)–(19)

The main result of this paper is predicated on the functions and parameters of equations (9) and (10) satisfying assumptions (11)–(19) listed in table 1. Whilst physical interpretations may be assigned to these assumptions, it is important to stress that these assumptions only represent *sufficient* conditions for the main result to hold. No assertions regarding *necessity* are made here. Consequently, such physical interpretations cannot be regarded as definitive requirements for the main result to hold, as *necessary and sufficient* conditions required for a proof could well be weaker than the stated assumptions (11)–(19). In any case, a physical description of these assumptions follows:

- Both saturation θ and photon relaxation Υ_1 should (strictly) increase with photon density, as stated in assumptions (11) and (12). Unlike photon relaxation, saturation can roll off at higher photon densities, although this is not a requirement. In comparative terms, photon relaxation should always exceed saturation, both in magnitude and in normalized growth rate, as stated in assumptions (13) and (14) respectively.

- Gain ϕ should (strictly) increase with carrier density without rolling off, as stated in assumption (15). For low carrier densities, the gain should be negative, and should increase through zero at some transparency carrier density, as stated in assumption (16).
- Non-radiative carrier relaxation Υ_2 should not (strictly) decrease with increasing carrier density at any point, as stated in assumption (17).
- Radiative carrier relaxation ψ should (strictly) increase with carrier density without rolling off, as stated in assumption (18).
- Spontaneous emission must occur, with some fraction of the emitted photons entering the lasing mode, as stated in assumption (19).

Continuous differentiability of all functions listed in table 1 is required to ensure existence of the corresponding derivatives and to guarantee local existence of solutions (via Lipschitz continuity) for the generic model (20).

4. Main result

4.1. Behavioural properties

The main result concerning the behaviour of the generic model (20) will be stated in terms of three behavioural properties, *well-posedness*, *input-to-state stability* and *stationary-point stability*. These properties are defined as follows.

4.1.1. Well-posedness. Generic model $\mathcal{G}$ is well-posed if for any initial density vector $x_\circ \in \mathbf{R}^2_{>0}$ and any input forcing function $u : \mathbf{R}_{\geqslant 0} \mapsto \mathbf{R}_{>0}$ (measurable, with $\|u\|_\infty < \infty$), there exists a unique phase-space trajectory $x(\cdot)$ that solves the initial value problem defined by (20) for all time $t \geqslant 0$ and remains confined to the open positive orthant $\mathbf{R}^2_{>0}$.

4.1.2. Input-to-state stability. Generic model $\mathcal{G}$ is input-to-state stable [19] if there exist functions $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that

$$|x(t)| \leqslant \gamma\,(\|u\|_\infty) + \beta(|x_\circ|, t), \qquad x := \mathcal{G}_{x_\circ}[u]$$

for all initial density vectors $x_\circ \in \mathbf{R}^2_{>0}$, all inputs u (measurable, with $\|u\|_\infty < \infty$), and all times $t \geqslant 0$.

4.1.3. Stationary-point stability. Generic model $\mathcal{G}$ is stationary-point stable if for each constant forcing function $u(t) := U \in \mathbf{R}_{>0}$, $t \geqslant 0$, there exists a unique $\bar{x}_U \in \mathbf{R}^2_{>0}$ such that

$$0 = \limsup_{t\to\infty} |x(t) - \bar{x}_U|\,, \qquad x := \mathcal{G}_{x_\circ}[u]$$

for all $x_\circ \in \mathbf{R}^2_{>0}$. (SPS is a special case of the Cauchy gain property [22].)

4.2. Statement

The main result of this paper may be stated simply as follows:

The generic model of (20) *is*

(I) *well-posed;*
(II) *input-to-state stable;*
(III) *stationary-point stable.*

Table 2. Generic model functions for selected laser diode models.

Generic	Model 1	Model 2	Model 3
x_1	$S/(\Gamma N_t)$	$S/(\Gamma N_t)$	$S/(\Gamma N_t)$
x_2	N/N_t	N/N_t	N/N_t
$\theta(x_1)$	$\Gamma x_1(1+\epsilon\Gamma N_t x_1)^{-1}$	$\Gamma x_1(1+\epsilon\Gamma N_t x_1)^{-1}$	$\Gamma x_1(1+\epsilon\Gamma N_t x_1)^{-1}$
$\phi(x_2)$	$g_0(x_2-1)$	$g_0\ln(x_2)$	$g_0(x_2-1)$
$\Upsilon_1(x_1)$	x_1/τ_{ph}	x_1/τ_{ph}	x_1/τ_{ph}
$\Upsilon_2(x_2)$	0	$CN_t^3x_2^3$	$C_0x_2+N_t^2C_2x_2^3+(D_r/N_t)(N_tx_2)^{\frac{11}{2}}$
ζ	β	$F\beta(F\beta+1-\beta)^{-1}$	β
$\psi(x_2)$	x_2/τ_{sp}	$N_tBx_2^2(F\beta+1-\beta)$	$C_1N_tx_2^2$
u	$I/(eVN_t)$	J/N_t	$I/(eVN_t)$

4.3. Qualitative description

The main result simply states that (I) the general model always has a well-defined solution for the initial conditions and forcing functions of interest, (II) bounded forcing functions always give rise to ultimately bounded trajectories in phase space and (III) any constant forcing function defines a unique, globally attracting equilibrium for those phase-space dynamics.

4.4. Application to models 1–3

The proof of stability for any particular laser diode model of the form of (9) and (10) rests on whether assumptions (11)–(19) are satisfied by that model. For the cases considered in this paper, this is readily determined by simple inspection and consideration of the various terms appearing in table 2. One can immediately see that the saturation factor used in model 4 causes θ for that model to be a decreasing function for sufficiently large values of the argument, in contradiction to the requirement of (11). Inspection of the form of θ for the other models in table 2 shows that the use of the saturation factor $(1+\epsilon S)^{-1}$ produces a θ which satisfies all requirements of (11).

Assumptions (11)–(14) place restrictions on the physical parameters appearing in θ and Υ_1. In fact, all the models listed in table 2 can be readily shown to satisfy all the requirements of assumptions (11)–(14) provided that $\epsilon > 0, \Gamma > 0, N_t > 0$ and the applied current is positive, for all physically allowable values of the remaining parameters. The proof requires that there is some gain saturation and some non-zero threshold for gain, namely assumptions (11) and (16), in order to guarantee stability. It is interesting to note that similar observations have been made previously by appeal to the observed behaviour of numerical solutions to similar models [30]. The requirement $\Gamma > 0$ is physically essential since $\Gamma = 0$ implies that there is no physical overlap between the gain region and the lasing mode.

Model 2 differs from model 1 in the forms of ϕ, Υ_2, ζ and ψ. A comment on the dependence of ζ and ψ on the Purcell factor F is in order since the motivation for the study by Lau *et al* was to determine the feasibility of enhanced dynamics in a laser diode by control of the Purcell factor. Since F is the factor by which the spontaneous emission is either increased or decreased, it is evident that $0 < F < \infty$. Since β is defined as the fraction of the spontaneous emission which is coupled to the lasing mode, β must satisfy $0 < \beta < 1$. ψ and ζ satisfy the requirements of assumptions (18) and (19) and as a result there is no additional restriction on the values of F or β if stability is required.

4.5. The problem with models 4 and 5

It is important to note that models 4 and 5 reported in section 2 do not fit the generic model described by (20).

In the case of model 4, it may be noted that the corresponding rate equations are of the form of (9) and (10), with $x_1 = S/(\Gamma N_t)$, $x_2 = N/N_t$, $\theta(x_1) = \Gamma x_1(1 - \epsilon \Gamma N_t x_1)$, $\phi(x_2) = g_0(x_2 - 1)$, $\Upsilon_1(x_1) = x_1/\tau_{\rm ph}$, $\Upsilon_2(x_2) = 0$, $\zeta = \beta$, $\psi(x_2) = x_2/\tau_{\rm sp}$ and $u = I$. However, this selection is not compatible with the listed assumptions (11)–(19). In particular, it is clear that θ is not strictly increasing as required by assumption (11). Hence, the stability result presented in this paper does not apply to model 4. This does not mean that model 4 is unstable. We emphasize that the restrictions placed on the generic model are sufficient conditions for the stability result presented in this paper.

In the case of model 5, it may be observed that the attendant rate equations (7) and (8) do not fit the form of the generic model equations (9) and (10). In particular, the asymmetry of the gain terms in the equations for S and N and the dependence of the saturation term on the carrier density does not allow simultaneous factorization of the gain terms θ and ϕ and the relaxation terms Υ_1 and Υ_2 into functions of S and N only. Consequently, the stability result presented in this paper does not apply to model 5.

4.6. Other laser diode models

There are many other forms of the laser diode model that appear in the literature. The applicability of the main result to those models depends on whether those models can be considered as special cases of the generic model (20) presented here. This means that model in question must satisfy the structure of (9) and (10), with the functions and parameters thus defined satisfying assumptions (11)–(19). For example, models with time-delayed or distributed feedback (for example, [12, 13]) are not consistent with the form of (9) and (10). However, those models without such feedback often are. Consider, for example, the models studied in [23]. The first model considered there ([23], equations (1)–(3)) uses a saturation term θ of the form $(1-\epsilon S)S$ that is inconsistent with assumption (11) as per model 4 discussed above. However, that particular model is known to suffer from the existence of three solution regimes under dc operating conditions, and so cannot possibly be stationary-point stable. It is interesting to note that the later model considered there ([23], equations (11)–(13)) is consistent with assumptions (11)–(19). The conclusion there of a single solution regime is consistent with an application of the main result here. (Note that the particular model employs three radiative carrier relaxation terms, which corresponds to terms of the form of $\sum_{i=1}^{3} \zeta_i \psi_i(x_2)$ and $\sum_{i=1}^{3} \psi_i(x_2)$ in (9) and (10). The main result here may be easily extended to cover this slightly more general form of (9) and (10).)

5. Proof of main result

The proof of the main result is presented via the following basic steps:

(I) Well-posedness

- (i) Positive invariance of $\mathbf{R}^2_{>0}$;
- (ii) Local existence and uniqueness of solutions;
- (iii) Well-posedness.

(II) Input-to-state stability;

(III) Stationary point stability

(i) Existence of a unique stationary point;

(ii) Exclusion of periodic orbits;

(iii) Stationary-point stability.

5.1. Well-posedness

5.1.1. Positive invariance of $\mathbf{R}^2_{>0}$. As the right-hand side of (9) and (10) form a continuous mapping on $\mathbf{R}^2_{>0}$, any solution of the initial value problem associated with (20) must be absolutely continuous. Consequently, in order to show that the open positive orthant $\mathbf{R}^2_{>0}$ is positively invariant, it is sufficient to show that no such solution can cross the boundary $\partial\mathbf{R}^2_{\geqslant 0}$ of the positive orthant. To this end, let $L, c, \zeta \in \mathbf{R}_{>0}$ denote real constants as per assumptions (14), (16) and (19), respectively. Given any $u \in \mathbf{R}_{>0}$ fixed, define the sets $\mathcal{O}_1^\delta, \mathcal{O}_2^\delta \subset \mathbf{R}^2_{>0}$,

$$\mathcal{O}_1^\delta := \left\{ x \in \mathbf{R}^2_{>0} \middle| x_1 \leqslant \Upsilon_1^{-1}\left(\frac{\zeta\,\psi(\delta)}{1+\left(\frac{-\phi(\delta)}{L}\right)}\right), x_2 \geqslant \delta \right\}$$

$$\mathcal{O}_2^\delta := \left\{ x \in \mathbf{R}^2_{>0} \middle| x_2 \leqslant \delta \right\}$$

where $\delta \in (0, \delta^*]$ is arbitrary, and

$$\delta^* := \min\left\{\frac{c}{2}, (\Upsilon_2 + \psi)^{-1} \circ \left(\frac{1}{2}u\right)\right\}. \tag{21}$$

Here, assumptions (16)–(18) guarantee that $\delta^* \in \mathbf{R}_{>0}$ is well defined. Assumptions (12), (14)–(16), (18) and (19) in turn guarantee that sets $\mathcal{O}_{1,2}^\delta$ are both non-empty. For convenience, set

$$e_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad e_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and let $F : \mathbf{R}^2_{>0} \times \mathbf{R}_{>0} \mapsto \mathbf{R}^2$ denote the vector field defined by the right-hand side of the generic model equations (9) and (10),

$$F(x, u) := \begin{bmatrix} \theta(x_1)\phi(x_2) - \Upsilon_1(x_1) + \zeta\,\psi(x_2) \\ -\theta(x_1)\phi(x_2) - \Upsilon_2(x_2) - \psi(x_2) + u \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{22}$$

Then,

$$\begin{aligned} \min_{x\in\mathcal{O}_1^\delta, x_2\geqslant c} \langle F(x,u), e_1\rangle &= \min_{x\in\mathcal{O}_1^\delta, x_2\geqslant c} \{\theta(x_1)\phi(x_2) - \Upsilon_1(x_1) + \zeta\,\psi(x_2)\} \\ &\geqslant \min_{x\in\mathcal{O}_1^\delta} \{\theta(x_1)\phi(c) - \Upsilon_1(x_1) + \zeta\,\psi(c)\} \\ &\geqslant \zeta\,\psi(c) - \frac{\zeta\,\psi(\delta)}{1+\left(\frac{-\phi(\delta)}{L}\right)} \\ &> 0 \end{aligned} \tag{23}$$

as (21) and assumption (15) imply that $\phi(\delta) \leqslant \phi(\delta^*) < 0$ and $\psi(\delta) \leqslant \psi(\delta^*) < \psi(c)$. Note that assumptions (11), (12) and (16) have been applied implicitly here. By assumptions (13) and (14), $\Upsilon_1(s) > L\theta(s)$ for all $s \in \mathbf{R}_{>0}$, so that

$$\begin{aligned} \min_{x\in\mathcal{O}_1^\delta, x_2<c} \langle F(x,u), e_1\rangle &= \min_{x\in\mathcal{O}_1^\delta, x_2<c} \{-\theta(x_1)\,[-\phi(x_2)] - \Upsilon_1(x_1) + \zeta\,\psi(x_2)\} \\ &> \min_{x\in\mathcal{O}_1^\delta, x_2<c} \left\{-\Upsilon_1(x_1)\left(1 + \left(\frac{-\phi(x_2)}{L}\right)\right) + \zeta\,\psi(x_2)\right\} \end{aligned}$$

$$\geqslant \min_{x_2 \in [\delta, c)} \left\{ \zeta\,\psi(x_2) - \frac{\zeta\,\psi(\delta)}{1 + \left(\frac{-\phi(\delta)}{L}\right)} \left(1 + \left(\frac{-\phi(x_2)}{L}\right)\right)\right\}$$
$$\geqslant \zeta\,\psi(\delta) - \zeta\,\psi(\delta) = 0. \tag{24}$$

Hence, combining (23) and (24),

$$\min_{x \in \mathcal{O}_1^\delta} \langle F(x,u), e_1\rangle > 0. \tag{25}$$

Similarly, with $u \in \mathbf{R}_{>0}$, assumptions (11), (15)–(18) combined with the definition of $\mathcal{O}_2^\delta$ and (21) imply that

$$\begin{aligned} \min_{x \in \mathcal{O}_2^\delta} \langle F(x,u), e_2\rangle &= \min_{x \in \mathcal{O}_2^\delta} \{\theta(x_1)\,[-\phi(x_2)] - (\Upsilon_2 + \psi)\,(x_2) + u\} \\ &\geqslant \min_{x_1 \in \mathbf{R}_{>0}} \{\theta(x_1)\,[-\phi(\delta)] - (\Upsilon_2 + \psi)\,(\delta) + u\} \\ &\geqslant \min_{x_1 \in \mathbf{R}_{>0}} \left\{\theta(x_1)\,[-\phi(\delta)] + \tfrac{1}{2}u\right\} \\ &> 0. \end{aligned} \tag{26}$$

Let $\mathcal{B}_\delta$ denote the boundary of $\mathcal{O}_1^\delta \cup \mathcal{O}_2^\delta$ that resides in the open positive orthant $\mathbf{R}^2_{>0}$,

$$\mathcal{B}_\delta := \left(\partial\mathcal{O}_1^\delta \cup \partial\mathcal{O}_2^\delta\right) \backslash \partial\mathbf{R}^2_{\geqslant 0} \tag{27}$$

and let n_δ denote the set-valued map,

$$n_\delta(x) := \begin{cases} \{e_1\} & x \in \mathcal{O}_1^\delta \backslash \mathcal{O}_2^\delta \\ \{e_1, e_2\} & x \in \mathcal{O}_1^\delta \cap \mathcal{O}_2^\delta \\ \{e_2\} & x \in \mathcal{O}_2^\delta \backslash \mathcal{O}_1^\delta. \end{cases}$$

When evaluated at any $x \in \mathcal{B}_\delta$, $n_\delta(x)$ defines a unit normal to boundary $\mathcal{B}_\delta$, pointing into the interior of $\mathbf{R}^2_{>0} \backslash \left(\mathcal{O}_1^\delta \cup \mathcal{O}_2^\delta\right)$. Hence, by application of (25) and (26),

$$u \in \mathbf{R}_{>0} \quad \Longrightarrow \quad \liminf_{\delta \to 0^+} \inf_{x \in \mathcal{B}_\delta} \min_{\eta \in n_\delta(x)} \langle F(x,u), \eta\rangle \geqslant 0.$$

That is, for positive inputs $u \in \mathbf{R}_{>0}$, should an absolutely continuous solution to (9) and (10) exist, the open positive orthant $\mathbf{R}_{>0}$ must be positively invariant with respect to the flow thus defined. That is,

$$x_\circ \in \mathbf{R}^2_{>0} \quad \text{and} \quad u : \mathbf{R}_{\geqslant 0} \mapsto \mathbf{R}_{>0} \quad \Longrightarrow \quad \mu(t, x_\circ; u) \in \mathbf{R}^2_{>0} \qquad \forall\, t \geqslant 0 \tag{28}$$

where $\mu(\cdot)$ denotes the flow of (9) and (10).

5.1.2. Local existence and uniqueness of solutions. Local existence and uniqueness of solutions can be demonstrated on finite time intervals using standard local Lipschitz arguments [21]. In particular, it is sufficient to show that the right-hand side of (9) and (10) defines a locally Lipschitz continuous mapping in x, uniformly in u on $\mathbf{R}^2_{>0} \times \mathbf{R}_{\geqslant 0}$. This may easily be demonstrated via [21] (lemma 2.3) and assumptions (11), (12), (15), (17) and (18). Given the standard semigroup and continuity properties [21] of trajectories $x(\cdot)$ associated with the generic model (20), the time horizons on which unique solutions exist locally may be concatenated and thus the time horizon of existence and uniqueness maximally extended (see [31], for example).

5.1.3. Well-posedness. With uniqueness of solutions inherited from the local domain, well-posedness boils down to the preclusion of finite escape times, where a finite escape time corresponds to a discontinuity of the second kind in the trajectory $x(\cdot)$ of (20). A finite escape time exists if, given an input u, there exists a $\bar{t} \in (0,\infty)$ such that $\lim_{t\to\bar{t}^-}|x(t)| = \infty$. Finite escape times may thus be excluded if for each bounded input u there exists a positively invariant bounded subset of $\mathbf{R}^2_{>0}$ to which all trajectories initialized in $\mathbf{R}^2_{>0}$ converge in finite time. Using arguments developed in the general framework of input-to-state stable systems [19, 32], it may be shown that such a subset exists, and hence a finite escape time cannot (see the argument preceding theorem 10.4.1 in [32]).

To this end, define the function $V : \mathbf{R}^2_{\geqslant 0} \mapsto \mathbf{R}_{\geqslant 0}$ on the open positive orthant by

$$V(x) := |x|_1 \tag{29}$$

where $|\cdot|_1$ denotes the 1-norm on $\mathbf{R}^2$. As the open positive orthant is positively invariant, the time derivative of $V(x(t))$ is well defined for each t for which the trajectory $x(t)$ of the generic model (20) exists. That is, (22) and (29) imply that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) &= \nabla_x V(x(t))\cdot F(x(t),u(t))\\ &= -\Upsilon_1(x_1(t)) - \Upsilon_2(x_2(t)) - (1-\zeta)\psi(x_2(t)) + u(t),\end{aligned} \tag{30}$$

where $x(\cdot)$ exists. Define the function $\gamma : \mathbf{R}_{\geqslant 0} \mapsto \mathbf{R}_{\geqslant 0}$ by

$$\gamma(s) := \min\left(\Upsilon_1\left(\frac{s}{2}\right), \Upsilon_2\left(\frac{s}{2}\right) + (1-\zeta)\,\psi\left(\frac{s}{2}\right)\right)$$

and note that $\gamma \in \mathcal{K}_\infty \cap C^1(\mathbf{R}_{>0})$ by assumptions (12) and (17)–(19). A triangle inequality holds for $\mathcal{K}_\infty$ functions, so that

$$\begin{aligned}\gamma(x_1+x_2) &\leqslant \gamma(2x_1) + \gamma(2x_2)\\ &\leqslant \min\left(\Upsilon_1(x_1), \Upsilon_2(x_1) + (1-\zeta)\,\psi(x_1)\right)\\ &\quad + \min\left(\Upsilon_1(x_2), \Upsilon_2(x_2) + (1-\zeta)\,\psi(x_2)\right)\\ &\leqslant \Upsilon_1(x_1) + \Upsilon_2(x_2) + (1-\zeta)\,\psi(x_2).\end{aligned} \tag{31}$$

Combining (29), (30) and (31) yields the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \leqslant -\gamma\left(V(x(t))\right) + u(t), \tag{32}$$

which holds wherever $x(\cdot)$ exists. Define the function $\chi : \mathbf{R}_{\geqslant 0} \mapsto \mathbf{R}_{\geqslant 0}$,

$$\chi(s) := \gamma^{-1}(2s). \tag{33}$$

Note that $\chi \in \mathcal{K}_\infty$ as $\gamma \in \mathcal{K}_\infty$. Hence, for any bounded input function $u : \mathbf{R}_{\geqslant 0} \mapsto \mathbf{R}_{>0}$ and any fixed $M \in [\|u\|_\infty, \infty)$, the following open set is well defined,

$$\Omega_M := \left\{x \in \mathbf{R}^2_{>0} \middle| V(x) < \chi(M)\right\}. \tag{34}$$

Suppose that $x_\circ \in \mathbf{R}^2_{>0}\setminus\Omega_M$. As $\|u\|_\infty \in (0, M]$,

$$\begin{aligned}-\gamma(V(x_\circ)) + u(0) &\leqslant -\gamma(V(x_\circ)) + \|u\|_\infty\\ &\leqslant -\gamma(V(x_\circ)) + \chi^{-1}(V(x_\circ))\\ &= -\tfrac{1}{2}\gamma(V(x_\circ)) < 0.\end{aligned} \tag{35}$$

In particular, if $x_\circ \in \partial\Omega_M := \overline{\Omega}_M \cap \mathbf{R}^2_{>0}$, (32) and (35) imply that $V(x(t)) < V(x_\circ) = \chi(M)$ for all $t \geqslant 0$. As $\mathbf{R}^2_{>0}$ is positively invariant, this implies that $\Omega_M \subset \mathbf{R}^2_{>0}$ is also positively invariant. Consequently, as Ω_M is bounded, any trajectory $x(\cdot)$ initialized at $x_\circ \in \Omega_M$ cannot escape to infinity.

In general, for any $x_\circ \in \mathbf{R}^2_{>0} \setminus \Omega_M$, (32) and (35) imply that $|x(t)|_1 = V(x(t)) < V(x_\circ) = |x_\circ|_1$ for all $t \geqslant 0$ such that $x(t) \notin \Omega_M$. That is, any trajectory $x(\cdot)$ initialized at $x_\circ \in \mathbf{R}^2_{>0} \setminus \Omega_M$ also cannot escape to infinity. Indeed, as it is also the case that for $x(t) \notin \Omega_M$,

$$-\gamma\,(V(x(t))) + u(t) \leqslant -M < 0,$$

integration of the inequality (32), together with positive invariance of Ω_M, implies that any such trajectory must enter Ω_M at some time $t_1 \leqslant \frac{|x_\circ|_1 - \frac{1}{2}\chi(M)}{M} < \infty$ and never leave.

5.2. *Input to state stability*

ISS follows as a direct consequence of (29) and (32), see for example [19].

5.3. *Stationary point stability*

5.3.1. Existence of a unique stationary point. For a constant input function $u(t) \equiv U \in \mathbf{R}_{>0}$ for all $t \geqslant 0$, any stationary point X of the dynamics of (20) must satisfy the pair of algebraic equations defined by $0 = F(X, U)$, mapping F given by (22). In order to show that these equations define a unique stationary point for any $U \in \mathbf{R}_{>0}$, the approach taken generalizes that of [23]. In particular, the two aforementioned algebraic equations are recast in terms of two C^1 functions $Q_\pm : \mathbf{R}_{>0} \mapsto \mathbf{R}_{>0}$ for each $U \in \mathbf{R}_{>0}$, yielding

$$X_1 = Q_+(X_2), \qquad X_1 = Q_-(X_2).$$

By demonstrating that Q_+ is strictly increasing and Q_- is strictly decreasing, the existence of a unique stationary point X can be thus proved. To this end, for $U \in \mathbf{R}_{>0}$ fixed, define

$$\begin{aligned} f_1(x) &:= \theta\,(x_1)\,\phi(x_2) - \Upsilon_1\,(x_1) + \zeta\,\psi(x_2),\\ f_2(x) &:= -\Upsilon_1\,(x_1) - \Upsilon_2(x_2) - (1-\zeta)\,\psi(x_2) + U \end{aligned}$$

and note that

$$\begin{aligned} \frac{\partial f_1}{\partial x_1}(x) &= -\Delta(x), \qquad \Delta(x) := \Upsilon_1'(x_1) - \theta'(x_1)\phi(x_2)\\ \frac{\partial f_2}{\partial x_1}(x) &= -\Upsilon_1'(x_1) < 0. \end{aligned} \tag{36}$$

Here, the inequality follows by assumptions (11) and (13). In particular, note that

$$\Upsilon_1'(s) = (\Upsilon_1(s)/\theta(s))\,\theta'(s) + \theta(s)\,(\Upsilon_1/\theta)'\,(s) > 0 \qquad \forall\, s \in \mathbf{R}_{>0}. \tag{37}$$

To show that a similar inequality holds for the first of these partial derivatives, note that by assumptions (11), (12), (13), (18), (19),

$$\begin{aligned} \theta(x_1)\Delta(x) &= \theta(x_1)\Upsilon_1'(x_1) - \theta'(x_1)\,[\theta(x_1)\phi(x_2)]\\ &= \theta(x_1)\Upsilon_1'(x_1) - \theta'(x_1)\,[\Upsilon_1(x_1) - \zeta\,\psi(x_2)]\\ &= [\theta(x_1)]^2\,(\Upsilon_1/\theta)'\,(x_1) + \theta'(x_1)\zeta\,\psi(x_2) > 0 \end{aligned} \tag{38}$$

for all $x \in \mathbf{R}^2_{>0}$. Hence, $\Delta(x) > 0$ for all $x \in \mathbf{R}^2_{>0}$. With both partial derivatives thus sign definite on $\mathbf{R}^2_{>0}$, the implicit function theorem thus (implicitly) defines the functions $Q_\pm : \mathbf{R}_{>0} \mapsto \mathbf{R}_{>0}$ according to the corresponding equations $f_1(x) = 0 = f_2(x)$, so that

$$0 = \theta(Q_+(\xi))\phi(\xi) - \Upsilon_1\,(Q_+(\xi)) + \zeta\,\psi(\xi), \tag{39}$$

$$0 = -\Upsilon_1(Q_-(\xi)) - \Upsilon_2(\xi) - (1-\zeta)\psi(\xi) + U. \tag{40}$$

To deduce the nature of Q_- defined by (40), note that $\Upsilon_1^{-1} \in \mathcal{K}_\infty$ by assumption (12). Consequently (40) may be solved explicitly for $Q_-(\xi)$, yielding

$$Q_-(\xi) = \Upsilon_1^{-1}(U - (\Upsilon_2 + (1-\zeta)\psi)(\xi)). \tag{41}$$

Assumptions (17)–(19) imply that $(\Upsilon_2 + (1-\zeta)\psi)^{-1} \in \mathcal{K}_\infty$, so that inspection of (41) reveals that

$$Q_- : (0, (\Upsilon_2 + (1-\zeta)\psi)^{-1}(U)) \mapsto (0, \Upsilon_1^{-1}(U))$$

is a strictly decrescent function.

To deduce the nature of $Q_+(\xi)$, first note that continuous differentiability of θ, Υ_1 and ψ (by assumptions (11), (12) and (18)) may be combined with the continuous differentiability of Q_+ (from the implicit function theorem) to permit differentiation of (39) with respect to ξ. This yields

$$\begin{aligned} 0 &= \theta(Q_+(\xi))\phi'(\xi) - [\Upsilon_1'(Q_+(\xi)) - \theta'(Q_+(\xi))\,\phi(\xi)]Q_+'(\xi) + \zeta\,\psi'(\xi) \\ &= \theta(Q_+(\xi))\phi'(\xi) - \Delta(Q_+(\xi), \xi)Q_+'(\xi) + \zeta\,\psi'(\xi), \end{aligned}$$

where Δ is as defined in (36). Inequality (38) then yields the implication

$$\begin{bmatrix} Q_+(\xi) \\ \xi \end{bmatrix} \in \mathbf{R}^2_{>0} \quad \Longrightarrow \quad \Delta\,(Q_+(\xi), \xi) > 0 \quad \Longrightarrow \quad Q_+'(\xi) \geqslant 0. \tag{42}$$

With a view to applying this implication, note that for any $\xi \in (0, c]$, (39) is obviously equivalent to

$$\Upsilon_1(Q_+(\xi)) + [-\phi(\xi)]\theta(Q_+(\xi)) = \zeta\,\psi(\xi)$$

in which $-\phi(\xi) \geqslant 0$ by assumptions (15) and (16). Assumptions (11), (12), (18) and (19) then yield the additional implication

$$\xi \in (0, c] \quad \Longrightarrow \quad Q_+(\xi) \in (0, \Upsilon_1^{-1}(\zeta\,\psi(\xi))] \subset \mathbf{R}_{>0}. \tag{43}$$

Combining (42) and (43) yields

$$\xi \in (0, c] \quad \Longrightarrow \quad Q_+(\xi) > 0, \qquad Q_+'(\xi) \geqslant 0.$$

That is, Q_+ is positive and non-decreasing on $(0, c]$. This conclusion may be extended to $\mathbf{R}_{>0}$ via (42) by integration of $Q_+'(\xi)$ beyond $\xi = c$.

To complete the proof of existence of a unique stationary point for $U \in \mathbf{R}_{>0}$, note by (41) and (43), there exists a $\xi \in \mathbf{R}_{>0}$ sufficiently small such that $Q_+(\xi) < Q_-(\xi)$. Thus, as Q_- is strictly decrescent and Q_+ is non-decreasing, there exists a unique $\xi^* \in \mathbf{R}_{>0}$ such that $Q_+(\xi^*) = Q_-(\xi^*)$, thereby defining the unique stationary point

$$X := \begin{bmatrix} Q_+(\xi^*) \\ \xi^* \end{bmatrix} \equiv \begin{bmatrix} Q_-(\xi^*) \\ \xi^* \end{bmatrix} \in \mathbf{R}^2_{>0}.$$

5.3.2. Exclusion of closed orbits. It was shown in section 5.1 above that for bounded inputs there exists a bounded open set, denoted here by Ω, that is positively invariant and attracting, with any trajectory of the generic model (20) ultimately confined to Ω in finite time. Consequently, no closed orbits, or segments of closed orbits, can exist in $\mathbf{R}^2_{>0} \setminus \Omega$ for such bounded inputs. Hence, for a closed orbit to exist in $\mathbf{R}^2_{>0}$ in the presence of such bounded inputs, that orbit must be confined entirely to Ω. Here, it will be shown that such orbits cannot exist in Ω in the presence of *constant inputs*, thereby excluding the existence of closed orbits anywhere in $\mathbf{R}^2_{>0}$ for constant inputs. This will be achieved via application of *Dulac's*

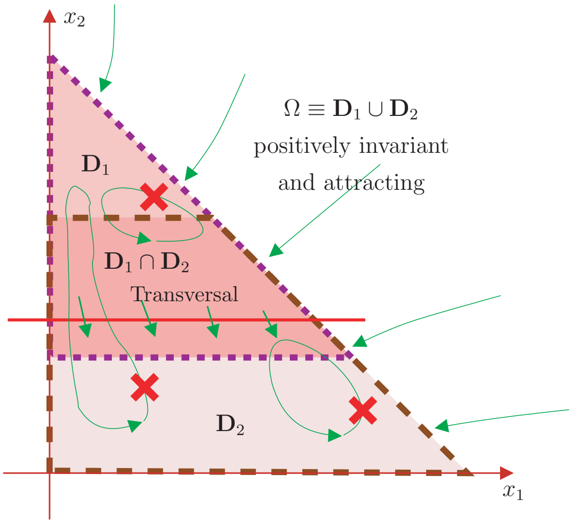

Figure 1. Excluding closed orbits from Ω.

criterion [20], which states the following:

> Consider the dynamical system $\dot{x} = f(x)$. Suppose there exists a continuously differentiable function $g : \mathbf{R}^2 \mapsto \mathbf{R}$ such that $\nabla \cdot (gf)$ is continuous and non-zero on some simply connected domain $\mathbf{D} \subset \mathbf{R}^2$. Then, no closed orbit can lie entirely in domain $\mathbf{D}$.

Here, we consider $f(x) := F(x, U)$, where F is defined by (22) and $U \in \mathbf{R}_{>0}$ is the constant input. $\Omega := \Omega_M \subset \mathbf{R}^2_{>0}$ is fixed by selecting a specific $M \in [U, \infty)$. Finding a function g such that the conditions of Dulac's criterion are satisfied for the dynamical system of (20) on the entirety of Ω is non-trivial. So, in order to exclude closed orbits from Ω, two overlapping domains $\mathbf{D}_1$ and $\mathbf{D}_2$ are considered, with $\mathbf{D}_1 \cup \mathbf{D}_2 \equiv \Omega$ and $\mathbf{D}_1\backslash\mathbf{D}_2$, $\mathbf{D}_2\backslash\mathbf{D}_1$ non-empty. By excluding closed orbits from lying entirely within $\mathbf{D}_1$ or entirely within $\mathbf{D}_2$, it follows that for a closed orbit to exist in $\mathbf{D}_1 \cup \mathbf{D}_2$, that closed orbit must repeatedly traverse $\mathbf{D}_1 \cap \mathbf{D}_2$ to visit the sets $\mathbf{D}_1\backslash\mathbf{D}_2$ and $\mathbf{D}_2\backslash\mathbf{D}_1$ *ad infinitum*. However, this possibility can be excluded by the construction of a transversal through $\mathbf{D}_1 \cap \mathbf{D}_2$, across which the flow is unidirectional. This argument is illustrated in figure 1, the details of which follow.

Define the constants $\hat{c}, \hat{M} \in \mathbf{R}_{>0}$ by

$$\hat{c} := \phi^{-1}\left(\frac{L}{2}\right), \tag{44}$$

$$\widehat{M} := \chi^{-1}(2\hat{c}) + U, \tag{45}$$

where L is defined by assumption (14), $\hat{c} > c > 0$ exists by assumptions (15) and (16), $U \in \mathbf{R}_{>0}$ is the constant input applied to (20), and χ is defined by (33). Note that $\widehat{M} > U$. Using these constants, define the open sets $\Omega, \mathbf{D}_1, \mathbf{D}_2 \subset \mathbf{R}^2_{>0}$ by

$$\Omega := \Omega_{\widehat{M}}, \quad \Omega_{(\cdot)} \text{ defined by (34)}, \tag{46}$$

$$\mathbf{D}_1 := \{x \in \Omega \mid x_2 > c\}, \tag{47}$$

$$\mathbf{D}_2 := \{x \in \Omega \,|\, x_2 < \hat{c}\}\,, \tag{48}$$

where it is obvious that $\mathbf{D}_1 \cup \mathbf{D}_2 \equiv \Omega$ as $c < \hat{c}$. It is also straightforward to show that the sets $\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_1\backslash\mathbf{D}_2, \mathbf{D}_2\backslash\mathbf{D}_1$ and $\mathbf{D}_1 \cap \mathbf{D}_2$ are all non-empty. For example, selecting $x \in \left(\frac{\hat{c}}{4}, \frac{5\hat{c}}{4}\right)$, (29) and (45) imply that $V(x) < 2\hat{c} < \chi(\widehat{M})$, so that $x \in \mathbf{D}_1\backslash\mathbf{D}_2$. In order to apply Dulac's criterion in $\mathbf{D}_1$ and $\mathbf{D}_2$ with F as defined in (22), first define

$$\begin{aligned} d(x,U) &:= \nabla \cdot [g(x)F(x,U)] \\ &= g(x)\,[\nabla \cdot F(x,U)] + [\nabla g(x)] \cdot F(x,U) \\ &= g(x)[\theta'(x_1)\phi(x_2) - \theta(x_1)\phi'(x_2) - \Upsilon_1'(x_1) - \Upsilon_2'(x_2) - \psi'(x_2)] \\ &\quad + \frac{\partial g}{\partial x_1}(x)\,[\theta(x_1)\phi(x_2) - \Upsilon_1(x_1) + \zeta\,\psi(x_2)] \\ &\quad + \frac{\partial g}{\partial x_2}(x)\,[-\theta(x_1)\phi(x_2) - \Upsilon_2(x_2) - \psi(x_2) + U]\,. \end{aligned} \tag{49}$$

Implicit in this definition is the application of assumptions (11), (12), (15), (17) and (18). Note that if $g : \mathbf{R}^2_{>0} \mapsto \mathbf{R}_{>0}$ is restricted to positive mappings, (49) implies that

$$\begin{aligned} \frac{d(x,U)}{g(x)} &= \left[\theta'(x_1) + \frac{\theta(x_1)}{g(x)}\frac{\partial g}{\partial x_1}(x)\right]\phi(x_2) - \left[\Upsilon_1'(x_1) + \frac{\Upsilon_1(x_1)}{g(x)}\frac{\partial g}{\partial x_1}(x)\right] \\ &\quad - \left[\theta(x_1)\phi'(x_2) + \Upsilon_2'(x_2) + \psi'(x_2) - \zeta\frac{\partial g}{\partial x_1}(x)\psi(x_2)\right] \\ &\quad + \left[\frac{1}{g(x)}\frac{\partial g}{\partial x_2}(x)\,[-\theta(x_1)\phi(x_2) - \Upsilon_2(x_2) - \psi(x_2) + U]\right]. \end{aligned} \tag{50}$$

Exclusion of entire orbits from $\mathbf{D}_1$: select $g : \mathbf{R}^2_{>0} \mapsto \mathbf{R}_{>0}$ as

$$g(x) = g_1(x_1) := \frac{1}{\Upsilon_1(x_1) + \theta(x_1)}$$

and note that g is well-defined on $\mathbf{R}^2_{>0}$. Observe that for all $x \in \mathbf{R}^2_{>0}$, assumptions (11) and (12) imply that

$$\frac{\partial g}{\partial x_1}(x) = -\,[g_1(x_1)]^2\,(\Upsilon_1'(x_1) + \theta'(x_1)) < 0, \qquad \frac{\partial g}{\partial x_2}(x) = 0$$

so that third and fourth bracketed terms in (50) are respectively positive and zero on $\mathbf{R}^2_{>0}$ by assumptions (11), (12), (15) and (17)–(19). Hence, rewriting the first two (remaining) bracketed terms of (50) yields the inequality

$$\frac{d(x,U)}{g(x)} < \frac{1}{g_1(x_1)}[(g_1\theta)'(x_1)\phi(x_2) - (g_1\Upsilon_1)'(x_1)].$$

Note that

$$\begin{aligned} \frac{(g_1\theta)'}{g_1} &= -\frac{\theta^2}{\Upsilon_1 + \theta}\left(\frac{\theta\Upsilon_1' - \Upsilon_1\theta'}{\theta^2}\right) = -g_1\theta^2\,(\Upsilon_1/\theta)'\,, \\ \frac{(g_1\Upsilon_1)'}{g_1} &= \frac{\theta^2}{\Upsilon_1 + \theta}\left(\frac{\theta\Upsilon_1' - \Upsilon_1\theta'}{\theta^2}\right) = g_1\theta^2\,(\Upsilon_1/\theta)'\,. \end{aligned}$$

Consequently,

$$\frac{d(x,U)}{g(x)} < -g_1(x)\,[\theta(x_1)]^2\,(\Upsilon_1/\theta)'\,(x_1)\,[\phi(x_2) + 1] < 0 \tag{51}$$

for all $x_1 > 0$ and $x_2 > c$, where assumptions (11)–(13), (15) and (16) have been applied. Hence, by definition (47), it is clear from (51) and Dulac's criterion that no closed orbits can exist entirely within $\mathbf{D}_1$.

Exclusion of entire orbits in domain $\mathbf{D}_2$: select $g : \mathbf{R}^2_{>0} \mapsto \mathbf{R}_{>0}$ to be any positive constant. Equation (50) yields

$$\begin{aligned}\frac{d(x,U)}{g(x)} &= \left[\theta'(x_1)\phi(x_2) - \frac{1}{2}\Upsilon_1'(x_1)\right] \\ &\quad - \left[\theta(x_1)\phi'(x_2) + \frac{1}{2}\Upsilon_1'(x_1) + \Upsilon_2'(x_2) + \psi'(x_2)\right] \\ &< \theta'(x_1)\phi(x_2) - \frac{1}{2}\Upsilon_1'(x_1)\end{aligned}$$

by assumptions (11), (15), (17) and (18). Note that assumptions (11)–(14) imply that

$$\Upsilon_1' > \left(\frac{\Upsilon_1}{\theta}\right)\theta' > L\theta'$$

so that

$$\frac{d(x,U)}{g(x)} < \theta'(x_1)\left[\phi(x_2) - \frac{L}{2}\right] < 0 \tag{52}$$

for all $x_1 > 0$ and $x_2 < \hat{c}$, where $\hat{c}$ is as per (44). Hence, by definition (48), it is clear from (52) and Dulac's criterion that no closed orbits can exist entirely within $\mathbf{D}_2$.

A transversal in $\mathbf{D}_1 \cap \mathbf{D}_2$: in order to construct a transversal for the dynamics of (20) in $\mathbf{D}_1 \cap \mathbf{D}_2$, first define $\hat{n} := \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ and $I_X : \mathbf{R}_{>0} \times \mathbf{R}_{>0} \mapsto \mathbf{R}$, $X \in \mathbf{R}_{>0}$ fixed, by

$$\begin{aligned}I_X(\xi, U) &:= \left\langle \hat{n}, F\left(\begin{bmatrix}\xi\\X\end{bmatrix}, U\right)\right\rangle \\ &= -\theta(\xi)\phi(X) - \Upsilon_2(X) - \psi(X) + U.\end{aligned}$$

With $\hat{c}$ defined as per (44), boundedness of Ω along with assumptions (11) and (15)–(18) guarantee the existence of the following positive constants:

$$K := \sup_{\xi \in \mathbf{R}_{>0}} \left\{\theta(\xi) \,\middle|\, \begin{bmatrix}\xi\\ \cdot\end{bmatrix} \in \Omega\right\}, \tag{53}$$

$$\delta^* := [\Upsilon_2 + \psi + K\phi]^{-1} \circ [\Upsilon_2 + \psi]\left(c + \frac{\hat{c} - c}{4}\right) - c, \tag{54}$$

$$U^* := [\Upsilon_2 + \psi + K\phi](c + \delta^*), \tag{55}$$

$$X_2^* := c + \frac{\delta^*}{2}, \tag{56}$$

$$\hat{X}_2^* := \hat{c} - \frac{\delta^*}{2}, \tag{57}$$

where $K, c \in \mathbf{R}_{>0}$. As $\hat{c} > c$ by assumptions (15) and (16),

$$\delta^* < [\Upsilon_2 + \psi + K\phi]^{-1} \circ [\Upsilon_2 + \psi + K\phi]\left(c + \frac{\hat{c} - c}{4}\right) - c = \frac{\hat{c} - c}{4}$$

$$\begin{aligned}\delta^* &> [\Upsilon_2 + \psi + K\phi]^{-1} \circ [\Upsilon_2 + \psi](c) - c \\ &= [\Upsilon_2 + \psi + K\phi]^{-1} \circ [\Upsilon_2 + \psi + K\phi](c) - c = 0\end{aligned}$$

so that

$$\delta^* \in \left(0, \frac{\hat{c} - c}{4}\right). \tag{58}$$

It is then straightforward to show that $x_2 = X_2^*$ is a transversal for the dynamics of (20) when $U \geqslant U^*$, as for all $\xi \in \mathbf{R}_{>0}$ such that $\left[\begin{smallmatrix}\xi \\ X_2^*\end{smallmatrix}\right] \in \Omega$, (53) and assumptions (15), (17) and (18) imply that

$$\begin{aligned} U \geqslant U^* \quad \Longrightarrow \quad I_{X_2^*}(\xi, U) &= -\theta(\xi)\phi(X_2^*) - \Upsilon_2(X_2^*) - \psi(X_2^*) + U \\ &\geqslant -[\Upsilon_2 + \psi + K\phi]\left(c + \frac{\delta^*}{2}\right) + [\Upsilon_2 + \psi + K\phi](c + \delta^*) \\ &> 0. \end{aligned}$$

Similarly, it is straightforward to show that $x_2 = \hat{X}_2^*$ is a transversal for the dynamics of (20) when $U < U^*$, as for all $\xi \in \mathbf{R}_{>0}$ such that $\left[\begin{smallmatrix}\xi \\ \hat{X}_2^*\end{smallmatrix}\right] \in \Omega$, assumptions (11), (12), (15), (16)–(18) imply that

$$\begin{aligned} U < U^* \quad \Longrightarrow \quad I_{\hat{X}_2}(\xi, U) &= -\theta(\xi)\phi(\hat{X}_2^*) - \Upsilon_2(\hat{X}_2^*) - \psi(\hat{X}_2^*) + U \\ &< -\theta(\xi)\phi(\hat{X}_2^*) - [\Upsilon_2 + \psi]\left(\hat{c} - \frac{\delta^*}{2}\right) \\ &\quad + [\Upsilon_2 + \psi + K\phi](c + \delta^*) \\ &= -\theta(\xi)\phi(\hat{X}_2^*) - [\Upsilon_2 + \psi]\left(\hat{c} - \frac{\delta^*}{2}\right) \\ &\quad + [\Upsilon_2 + \psi]\left(c + \frac{\hat{c} - c}{4}\right) \\ &< -\theta(\xi)\phi(\hat{X}_2^*) - [\Upsilon_2 + \psi]\left(c + \frac{\hat{c} - c}{4}\right) \\ &\quad + [\Upsilon_2 + \psi]\left(c + \frac{\hat{c} - c}{4}\right) < 0. \end{aligned}$$

Note that both transversals are contained entirely within $\mathbf{D}_1 \cap \mathbf{D}_2$ by (56)–(58). Hence, it follows that no closed orbits can cross $\mathbf{D}_1 \cap \mathbf{D}_2$. As there can be no closed orbits entirely within $\mathbf{D}_1$, entirely within $\mathbf{D}_2$, or that cross $\mathbf{D}_1 \cap \mathbf{D}_2$, it follows that no closed orbits can lie within $\mathbf{D}_1 \cup \mathbf{D}_2 \equiv \Omega$. As no closed orbits or segments of closed orbits can exist in $\mathbf{R}^2_{>0} \setminus \Omega$, it follows that no closed orbits can exist in $\mathbf{R}^2_{>0}$ as a result of the constant input $U \in \mathbf{R}_{>0}$ being applied to the generic model (20). As U is arbitrary here, this conclusion also holds true for *any* constant input.

5.3.3. Stationary-point stability. As the phase space of (20) is two dimensional, the range of dynamical behaviours for constant inputs is restricted to sinks, sources, saddles and closed orbits [20]. Most notably, chaotic behaviour is not possible. Restricting attention to the positive orthant, as no closed orbit can exist, behaviour is necessarily limited to sinks, sources or saddles, each of which must be located at a stationary point. As there is only one stationary point in the positive orthant, only one of these behaviours can be exhibited. As the existence of an unstable manifold in the positive orthant is precluded by ISS, and such a manifold is implied by the existence of a saddle or source, it follows that the stationary point must indeed be a sink, which must be asymptotically stable (theorem 4.14, [20]). As all other behaviours are excluded, the domain of attraction for this sink must be the entire positive orthant.

6. Conclusion

A reasonably general rate equation based model was presented for capturing the structure of common laser diode models. It was shown that in the presence of reasonably general constraints on the functions defining this model, and on the currents that may be applied as inputs to that model, that physically reasonable stability properties may be guaranteed independently of the particular values of many of the parameters involved. In particular, stability of the standard model and Purcell enhanced model [2] may be verified generally, including all positive values of the Purcell factor F, whilst those utilizing $1 - \epsilon S$ saturation terms [24, 27] cannot.

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